Problem 1.31

In Section 1.5 we proved that Newton's third law implies the conservation of momentum. Prove the converse, that if the law of conservation of momentum applies to every possible group of particles, then the interparticle forces must obey the third law. [*Hint:* However many particles your system contains, you can focus your attention on just two of them. (Call them 1 and 2.) The law of conservation of momentum says that if there are no external forces on this pair of particles, then their total momentum must be constant. Use this to prove that $\mathbf{F}_{12} = -\mathbf{F}_{21}$.]

Solution

Suppose there's a system of N particles in space. Bring two of these particles far away from the rest so that the only interparticle forces to consider are the ones between these two. Let $\mathbf{F}_{12}$ be the force acting on particle 1 from particle 2, and let $\mathbf{F}_{21}$ be the force acting on particle 2 from particle 1. Assume that the law of conservation of momentum is true and that there are no external forces acting on these two particles. Then the sum of their momenta is a constant.

$$\mathbf{p}_1 + \mathbf{p}_2 = \text{constant}$$

Take the derivative of both sides with respect to time.

$$\frac{d}{dt}(\mathbf{p}_1 + \mathbf{p}_2) = \frac{d}{dt}(\text{constant})$$

$$\frac{d\mathbf{p}_1}{dt} + \frac{d\mathbf{p}_2}{dt} = 0$$

According to Newton's second law, the rate of change of a particle's momentum is equal to the sum of the forces acting on the particle.

$$\left(\sum \mathbf{F}\right)_1 + \left(\sum \mathbf{F}\right)_2 = 0$$

$$\mathbf{F}_{12} + \mathbf{F}_{21} = 0$$

Therefore,

$$\mathbf{F}_{12} = -\mathbf{F}_{21}.$$